1-Apr Lecture 15

Wednesday, April 1, 2026 7:47 AM

problem from last time: solve 2 dimensional system

$$x' = -2y$$
$$y' = \frac{1}{2}x$$

$$(x' = -2y)'$$
$$x'' = -2y'$$
$$x'' = -2\left(\frac{1}{2}x\right)$$

$$x'' = -x \Rightarrow x'' + x = 0$$

char eqn: $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$\therefore x(t) = A\cos t + B\sin t$$

introduce constants

let $A = C\cos\alpha$ let $B = C\sin\alpha$

$$= C\cos\alpha\cos t + C\sin\alpha\sin t \leftarrow \text{cosine difference formula}$$

$$x(t) = C(\cos(t-\alpha))$$

next, utilize $x' = -2y$ to determine $y(t)$

$$y = -\frac{1}{2}x'$$
$$y(t) = -\frac{1}{2}x'(t)$$
$$= -\frac{1}{2}C(\cos(t-\alpha))'$$
$$= -\frac{1}{2}C(-\sin(t-\alpha))$$
$$y(t) = \frac{1}{2}C\sin(t-\alpha)$$

square y:

Pyth. ID:
$\sin^2\theta + \cos^2\theta = 1$ let $\theta = t - \alpha$
$y^2 + x^2 = 1$

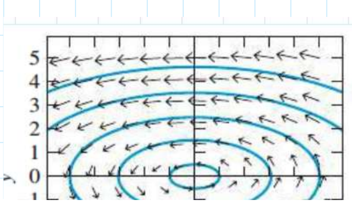
$$y^2 = \left(\frac{C}{2}\right)^2 \sin^2\theta$$

square x: $$x^2 = C^2\cos^2\theta$$

solve for $\sin^2\theta$, $\cos^2\theta$

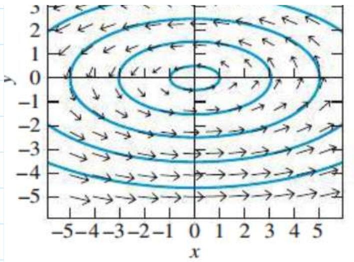
plug into Pyth. ID: $$\frac{y^2}{\left(\frac{C}{2}\right)^2} + \frac{x^2}{C^2} = 1$$

recall ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$



sol'n curves form ellipses w center @ $(0,0)$

a solution $(x(t), y(t))$ of a 2-dim. system



a solution $(x(t), y(t))$ of a 2-dim. system

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

may be regarded as a parameterization of a sol'n curve

continuing with this example ...

given init. conditions $x(0) = 2$ ← t

$$y(0) = 0$$

$x(t) = A\cos t + B\sin t$

$x(0) = A = 2$

$\therefore x(t) = 2\cos t$

$y(t) = -\frac{1}{2}x'$

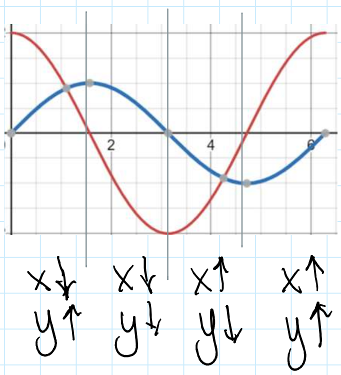
$= -\frac{1}{2}(-A\sin t + B\cos t)$

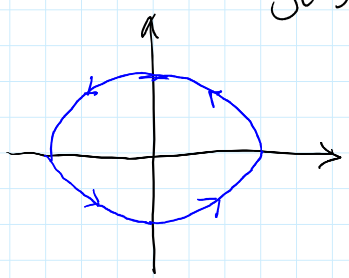
$y(0) = -\frac{1}{2}(\quad B) = 0$

$B = 0$

$\therefore y(t) = -\frac{1}{2}(-2\sin t)$

$y(t) = \sin t$





can say that, as t increases, the "sol'n point" of $(x(t), y(t))$ parameterizes an ellipse in CCW direction

ex. find general sol'n of system: $x' = y$

$$y' = 2x + y$$

differentiate

$x'' = y' \longrightarrow x'' = 2x + y$

$x'' = 2x + x'$ ← all x's

$x'' - x' - 2x = 0$

char eqn: $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \quad r_2 = -1$

$$(r-2)(r+1)=0$$
$$r_1=2 \quad r_2=-1$$
$$\boxed{\begin{aligned} x(t) &= Ae^{2t}+Be^{-t} \\ x' = y(t) &= 2Ae^{2t}-Be^{-t}\end{aligned}}$$

ex. Solve IVP $x'=-y$
$y'=(1.01)x-(0.2)y$ given $x(0)=0$; $y(0)=1$

$$x''=-y'$$
$$=-(1.01x-.2y)$$
$$=-1.01x+.2y$$
$$=.2y-1.01x$$
$$x''=-.2x'-1.01x$$
$$x''+.2x'+1.01x=0$$

Char eqn. $r^2+.2r+1.01=0 \Rightarrow r=-\frac{1}{10}\pm i$

$$x(t)=e^{-t/10}(A\cos t+B\sin t)$$

1^{st} init. cond: $x(0)= \quad A=0$

$$\therefore x(t)=e^{-t/10}B\sin t$$

since $x'=-y \Rightarrow y=-x'$

$$x(t)=Be^{-t/10}\sin t$$
$$x'(t)=-\tfrac{1}{10}Be^{-t/10}\sin t+Be^{-t/10}\cos t$$
$$y(t)=\tfrac{1}{10}Be^{-t/10}\sin t-Be^{-t/10}\cos t \leftarrow y=-x'$$

2^{nd} init cond. $y(0)= \quad -B=1 \Rightarrow B=-1$

$$\boxed{\begin{aligned} x(t)&=-e^{-t/10}\sin t \\ y(t)&=-\tfrac{1}{10}e^{-t/10}\sin t+e^{-t/10}\cos t\end{aligned}}$$

§4.2 Method of Elimination

similar to the Elimination Method in solving a linear system of algebraic eq'ns:

ex. $2x+y=7$
add $-2x+3y=12$

ex. $\quad 2x + y = 7$

add $\; -2x + 3y = 12$

etc.

ex. find solution of system: $\quad x' = 4x - 3y \quad \text{I}$

$\qquad\qquad\qquad\qquad\qquad\qquad y' = 6x - 7y \quad \text{II}$

given $x(0) = 2$; $y(0) = -1$

solve II for x : $\quad y' + 7y = 6x \Rightarrow \frac{1}{6}y' + \frac{7}{6}y = x \quad \text{III}$

then $x' = \frac{1}{6}y'' + \frac{7}{6}y' \quad \text{IV}$

substitute III, IV into I :

$$\frac{1}{6}y'' + \frac{7}{6}y' = \frac{4}{6}y' + \frac{14}{3}y - \frac{9}{3}y$$

$$6\left(\frac{1}{6}y'' + \frac{1}{2}y' - \frac{5}{3}y = 0\right)$$

$$y'' + 3y' - 10y = 0$$

$$r^2 + 3r - 10 = 0$$

$$(r+5)(r-2) = 0$$

$$r_1 = -5 \qquad r_2 = 2$$

$$\boxed{y(t) = c_1 e^{-5t} + c_2 e^{2t}}$$

$$y'(t) = -5c_1 e^{-5t} + 2c_2 e^{2t}$$

substitute y, y' into III

$$x = \frac{1}{6}y' + \frac{7}{6}y$$

$$x = \frac{1}{6}\left(-5c_1 e^{-5t} + 2c_2 e^{2t}\right) + \frac{7}{6}\left(c_1 e^{-5t} + c_2 e^{2t}\right)$$

$$= \underline{-\frac{5}{6}c_1 e^{-5t}} + \frac{1}{3}c_2 e^{2t} + \underline{\frac{7}{6}c_1 e^{-5t}} + \frac{7}{6}c_2 e^{2t}$$

$$\boxed{x(t) = \frac{1}{3}c_1 e^{-5t} + \frac{3}{2}c_2 e^{2t}}$$

apply initial conditions:

$$x(0) = \frac{1}{3}c_1 + \frac{3}{2}c_2 = 2$$

$$y(0) = c_1 + c_2 = -1 \Rightarrow c_2 = -1 - c_1$$

$$\frac{1}{3}c_1 + \frac{3}{2}(-1 - c_1) = 2$$

$$\frac{1}{3}c_1 - \frac{3}{2} - \frac{3}{2}c_1 = 2$$

$$-\frac{7}{6}c_1 \qquad\qquad \frac{4}{2} + \frac{3}{2}$$

$\frac{2}{2}\cdot\frac{1}{3} - \frac{3}{2}\cdot\frac{3}{3}$

$= \frac{2-9}{6} = -\frac{7}{6}$

$$-\frac{7}{6}c_1 = \frac{7}{2}$$

$$c_1 = -\frac{7}{2}\cdot\frac{6}{7} = -3$$

$$\therefore C_2 = -1+3 = 2$$

then desired sol'ns are:

$$x(t) = \frac{1}{3}(-3)e^{-5t} + \frac{3}{\cancel{2}}(\cancel{2})e^{2t}$$

$$y(t) =$$

$$\boxed{\begin{aligned} x(t) &= 3e^{2t} - e^{-5t} \\ y(t) &= 2e^{2t} - 3e^{-5t} \end{aligned}}$$

ex. find general sol'n of system: $x'' = -4x + \sin t$ I

$y'' = 4x - 8y$ II

rewrite I: $x'' + 4x = \sin t$ non HG

char eqn. $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$x_c = c_1\cos 2t + c_2 \sin 2t$$

$$x_p = A\sin t$$

$$x_p' = A\cos t$$

$$x_p'' = -A\sin t$$

plug into I: $x'' + 4x = \sin t$

$$-A\sin t + 4A\sin t = \sin t$$

$$3A\sin t = \sin t$$

$$3A = 1$$

$$A = \frac{1}{3}$$

$$x_p = \frac{1}{3}\sin t$$

$$x(t) = x_c + x_p$$

$$\boxed{x(t) = c_1\cos 2t + c_2 \sin 2t + \frac{1}{3}\sin t}$$

plug into II: $y'' = 4x - 8y$

$$y'' + 8y = 4c_1\cos 2t + 4c_2\sin 2t + \frac{4}{3}\sin t \quad \text{I}$$

$$r^2 + 8 = 0$$

$$r = \pm 2\sqrt{2}\,i$$

$$y_c = c_3\cos(2\sqrt{2}t) + c_4 \sin(2\sqrt{2}t)$$

$$y_p = B\cos 2t + C\sin 2t + D\sin t$$

$$y_p = B\cos 2t + C\sin 2t + D\sin t$$

$$y_p' = -2B\sin 2t + 2C\cos 2t + D\cos t$$

$$y_p'' = -4B\cos 2t - 4C\sin 2t - D\sin t$$

plug into V: $-4B\cos 2t - 4C\sin 2t - D\sin t + 8B\cos 2t + 8C\sin 2t + D\sin t = 4c_1\cos 2t + 4c_2\sin 2t + \frac{4}{3}\sin t$

$\cos 2t$: $-4B + 8B = 4c_1 \Rightarrow \frac{4B}{4} = c_1 \Rightarrow c_1 = B$

$\sin 2t$: $-4C + 8C = 4c_2$

$\frac{4C}{4} = c_2 \Rightarrow c_2 = C$

$\sin t$: $-D + 8D = \frac{4}{3}$

$7D = \frac{4}{3} \qquad D = \frac{4}{21}$

$y(t) = y_c + y_p$

$$\boxed{y(t) = c_3\cos(2\sqrt{2}t) + c_4\sin(2\sqrt{2}t) + c_1\cos 2t + c_2\sin 2t + \frac{4}{21}\sin t}$$